Analysis I Lecture 23

Last time:

Series

coefficients

center of the series

$$\boxed{\sum_{k=0}^{\infty} a_k (x - x_0)^k}$$

variable

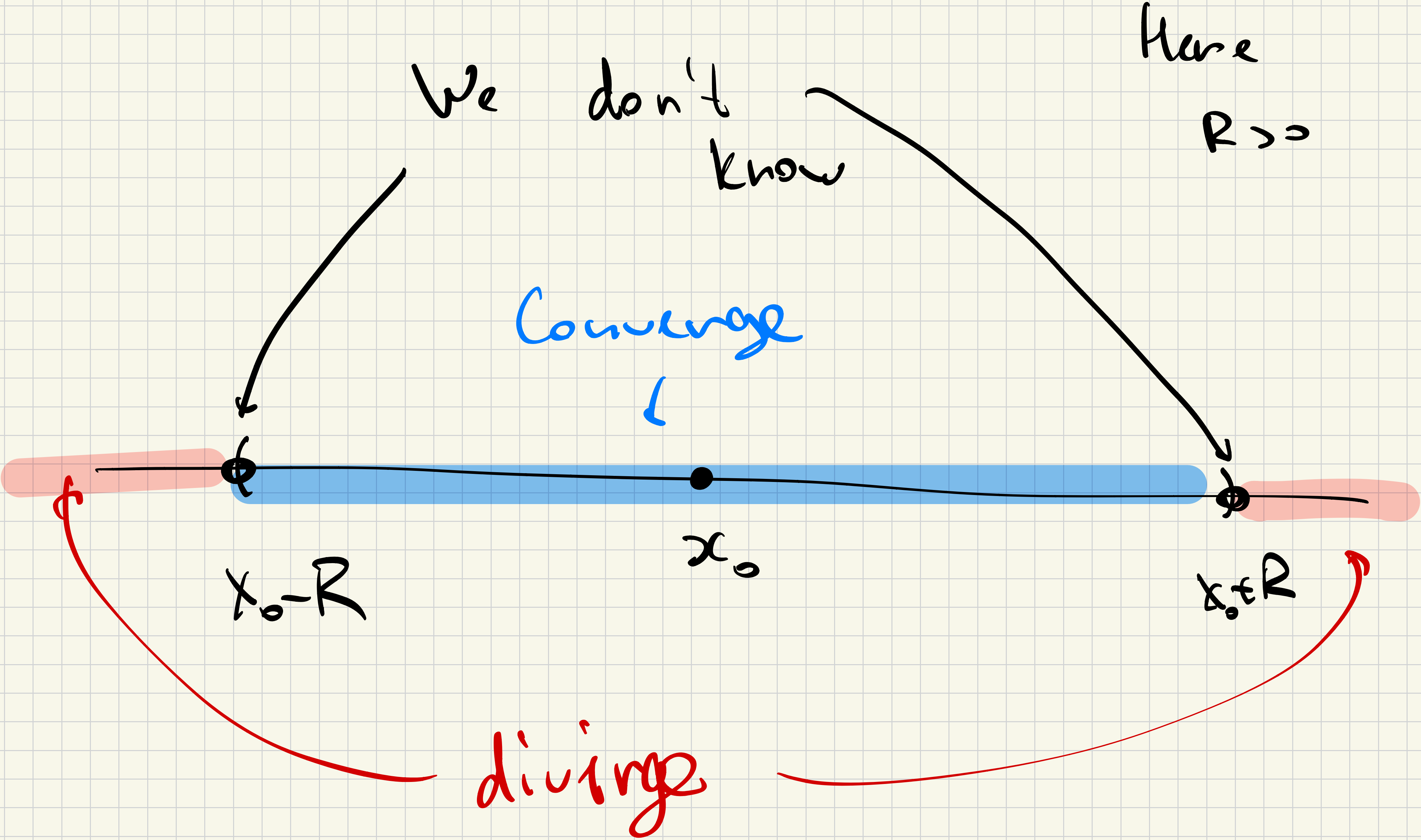
Theorem $\exists\ R \geq 0$ or $+\infty$

$\leftarrow$ radius of convergence

$$\sum_{k=0}^{\infty} a_k (x - x_0)$$

converges if $|x - x_0| < R$

diverges if $|x - x_0| > R$



We don't know
Here $R>0$
Converge
x_0
x_0-R
x_0+R
diverge

__Theorem__ Let $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ be a series

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell_1$ exists then $R = \frac{1}{\ell_1}$
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell_2$ exists then $R = \frac{1}{\ell_2}$.

__Here__ we write "$\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$"

Writing functions as power series

Let $f(x)$ be C^{∞}-function then the Taylor series of f is defined as

$$\boxed{T_{f,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}$$

When do we have that $f(x) = T_{f,x_0}(x)$?

Warning

Example: Consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Claim $f(x)$ is C^∞-function with

$$f^{(n)}(0) = 0 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad T_{f,0}(x) = \sum_{k=0}^{\infty} \frac{0}{k!} \cdot x^k = 0$$

But $f(x) \neq 0$ for $x \neq 0$.

So we get C^∞-function

s.t. $T_{f,0}$ converges everywhere

but $T_{f,0}(x) \neq f(x)$ for

$x \neq 0$.

<u>Definition</u> we will call a C^∞- function f analytic at x_0 if $\exists\ \varepsilon > 0$ s.t.

$$f(x) = T_{f, x_0}(x) \quad \text{for } |x - x_0| < \varepsilon.$$

The previous example shows that $f(x) = e^{-\frac{1}{x^2}}$ is not ANALYTIC at 0.

Proposition 9.13 Let f be C^∞ and let $T_f(x)$ be its Taylor series centered at 0

Then $f(x) = T_f(x)$ for $x > 0$ iff

$$\boxed{\lim_{n\to\infty} \left(\sup_{y \in (0,x)} |f^{(n)}(y)| \right) \frac{|x|^n}{n!} = 0}$$

for $x < 0$: $\lim_{n\to\infty} \left(\sup_{y \in (x,0)} |f^{(n)}(y)| \right) \frac{|x|^n}{n!} = 0$.

List of Taylor series of standard functions

$f(x)$	$T_{f,0}(x)$	radius of convergence of $T_{f,0}(x)$	$f(x) = T_{f,0}(x)$ over the interval
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	1	$]-1,1[$
$\frac{1}{1+x}$	$\sum_{k=0}^{\infty}(-1)^k x^k$	1	$]-1,1[$
e^x	$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$	$+\infty$	$\mathbb{R}$
$\log(1+x)$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$	1	$]-1,1[$
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$	$+\infty$	$\mathbb{R}$
$\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$	$+\infty$	$\mathbb{R}$
$\arctan(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$	1	$]-1,1[$
$\cosh(x)$	$\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$	$+\infty$	$\mathbb{R}$
$\sinh(x)$	$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$	$+\infty$	$\mathbb{R}$

<u>Example</u>

- $T_{f+g}(x) = T_f(x) + T_g(x)$

since $(f+g)^{(n)} = f^{(n)} + g^{(n)}$

- Let $T_{f,0}(x) = \sum_{n=0}^{\infty} a_n x^n$

Then $T_{f,0}(\lambda \cdot x)(x) = \sum_{n=0}^{\infty} \underbrace{\lambda^n \cdot a_n}_{=} x^n$

since $\left(f(\lambda x)\right)' = f'(\lambda x) \cdot (\lambda \cdot x)' =$
$= \lambda f'(\lambda x)$

$\left(f(\lambda x)\right)^{(n)} = \underline{\lambda^n} \cdot f^{(n)}(\lambda x)$.

$\underline{\text{E.g}}$. $f(x) = \underbrace{\frac{1}{1-2x} + e^x}$

$\underline{\text{Recall}}$

$$T_{\frac{1}{1-x},0} = \sum_{k=0}^{\infty} x^k$$

$$T_{e^x,0} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

To compute

$$\frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k$$

$$\Rightarrow T_{f,0} = \sum_{k=0}^{\infty} \left(2^k + \frac{1}{k!}\right) x^k$$

$e^x = \sum \frac{1}{k!} \cdot x^k$

Today: Integration

Two motivations:

1) Algebraic: Integration is inverse to differentiation.

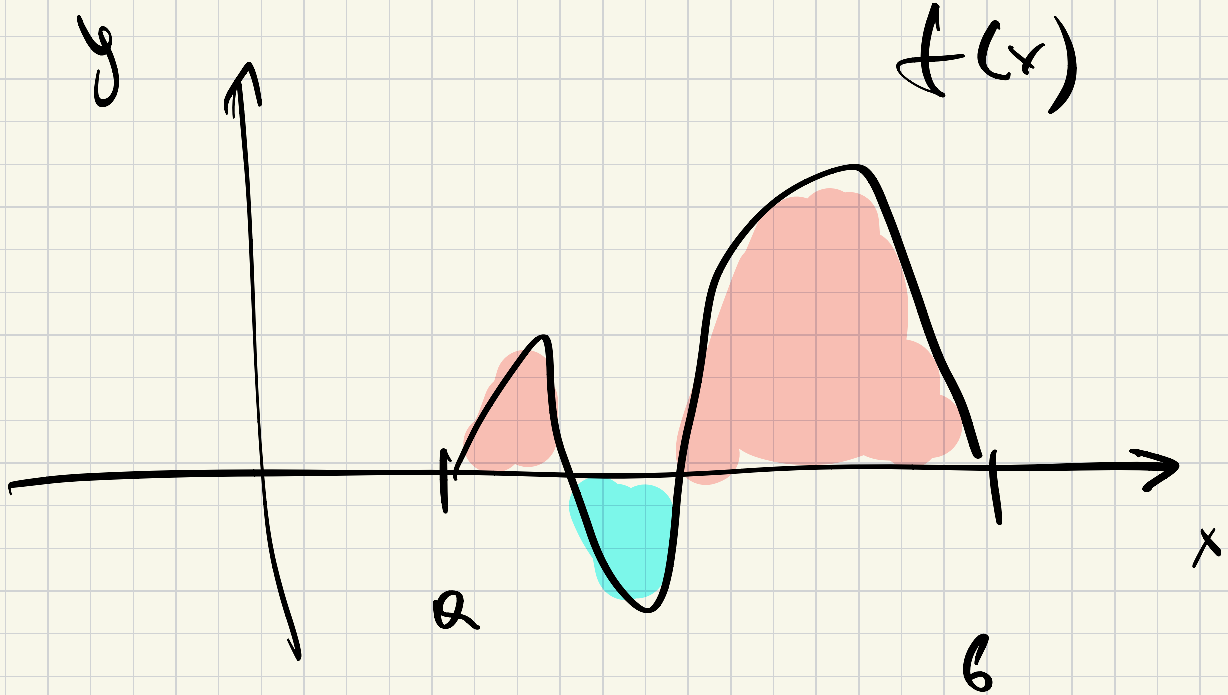
denote by $F(x) = \int_0^x f(y)\, dy$

called anti-derivative

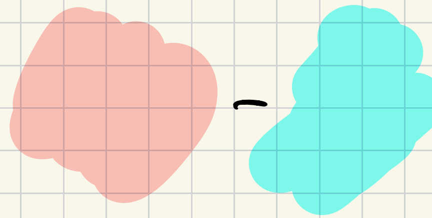
How to find $F(x)$ s.t. $F(x)' = f(x)$?

2)

Geometric: Area under the graph



$$\int_a^b f(x) =$$

$=$ 

Fundamental theorem of calculus:

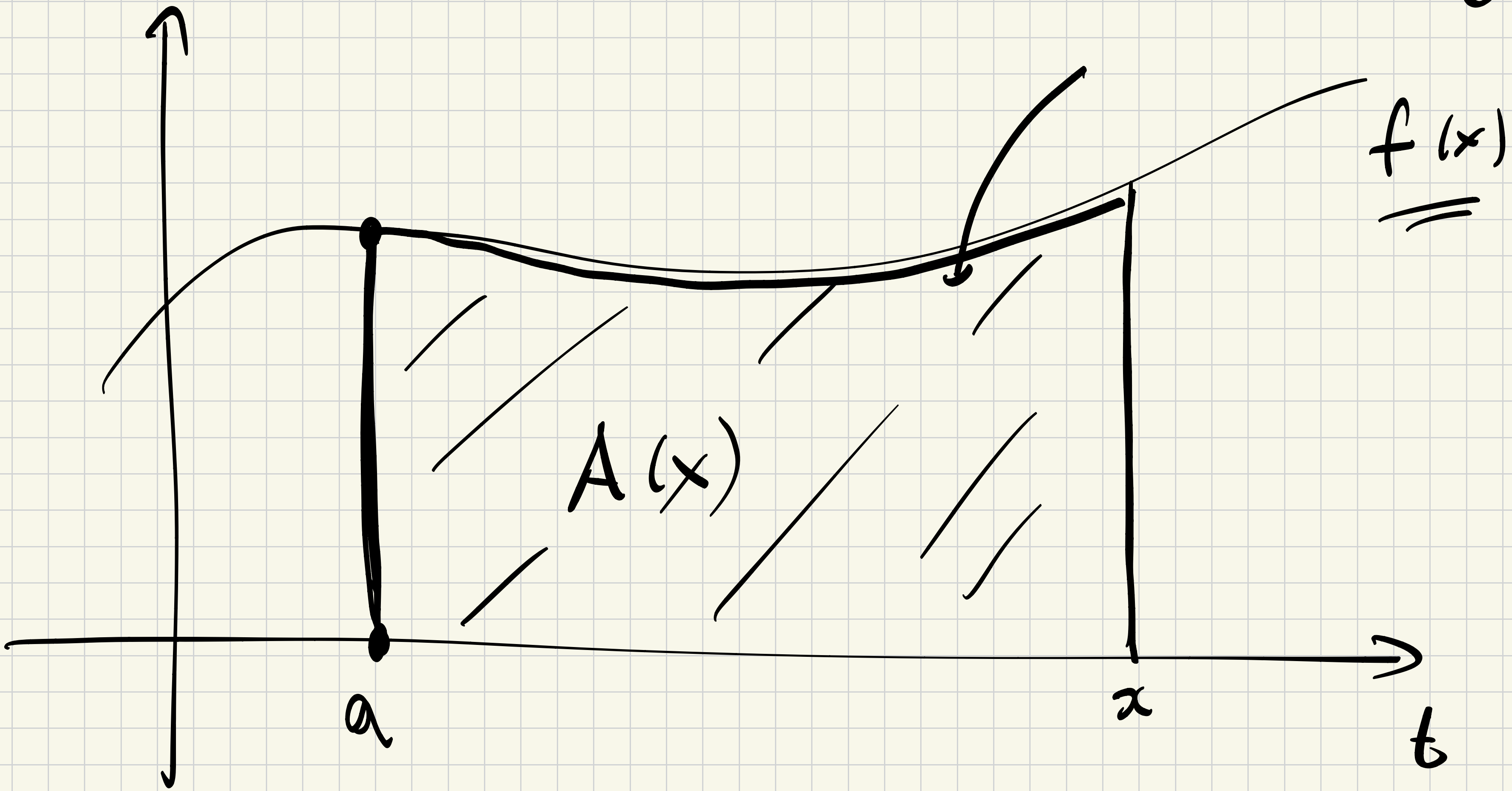
Two motivational questions have (almost) the same answer.

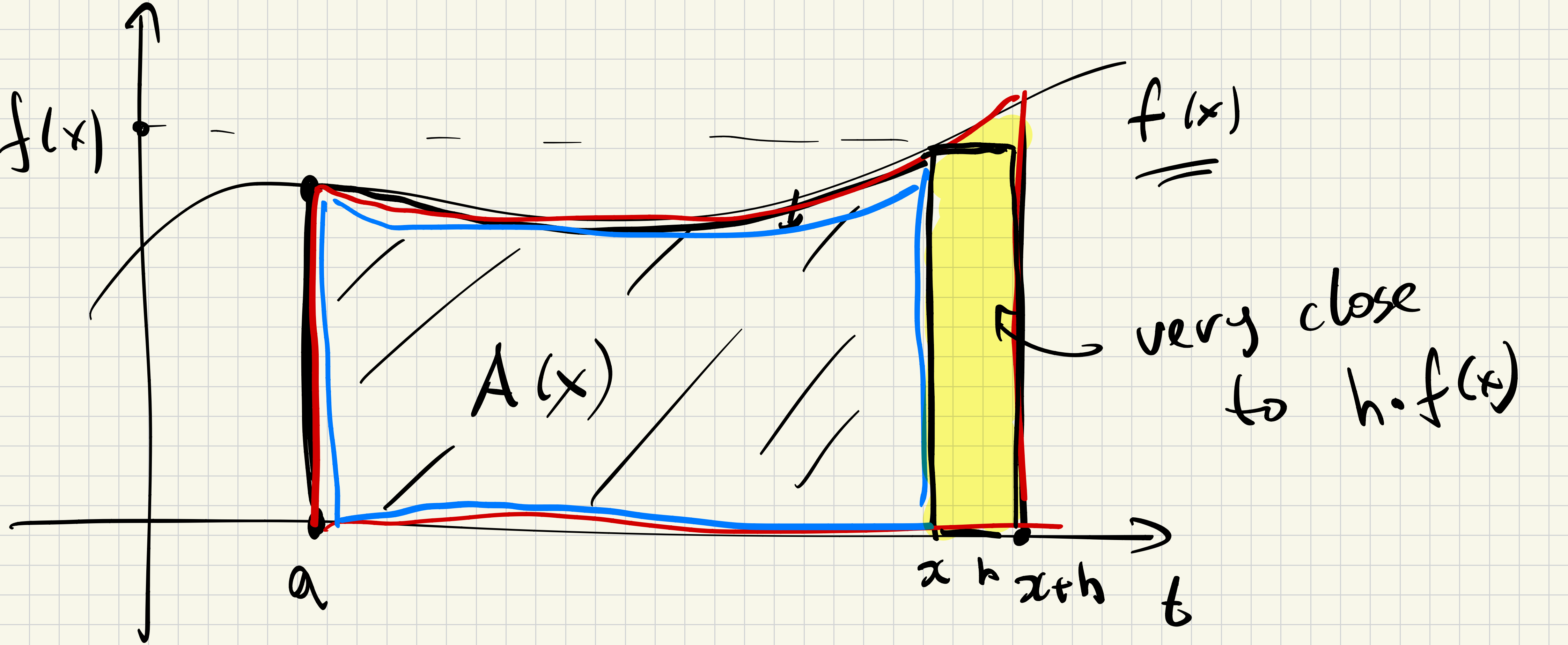
The anti-derivative is not unique:

- If $F' = f$ then $(F + c)' = f$ for any constant c
- But the area computation gives you concrete value.

Why FTC is true:

Define $A(x)$ = Area under the graph between $t=a$ and $t=x$

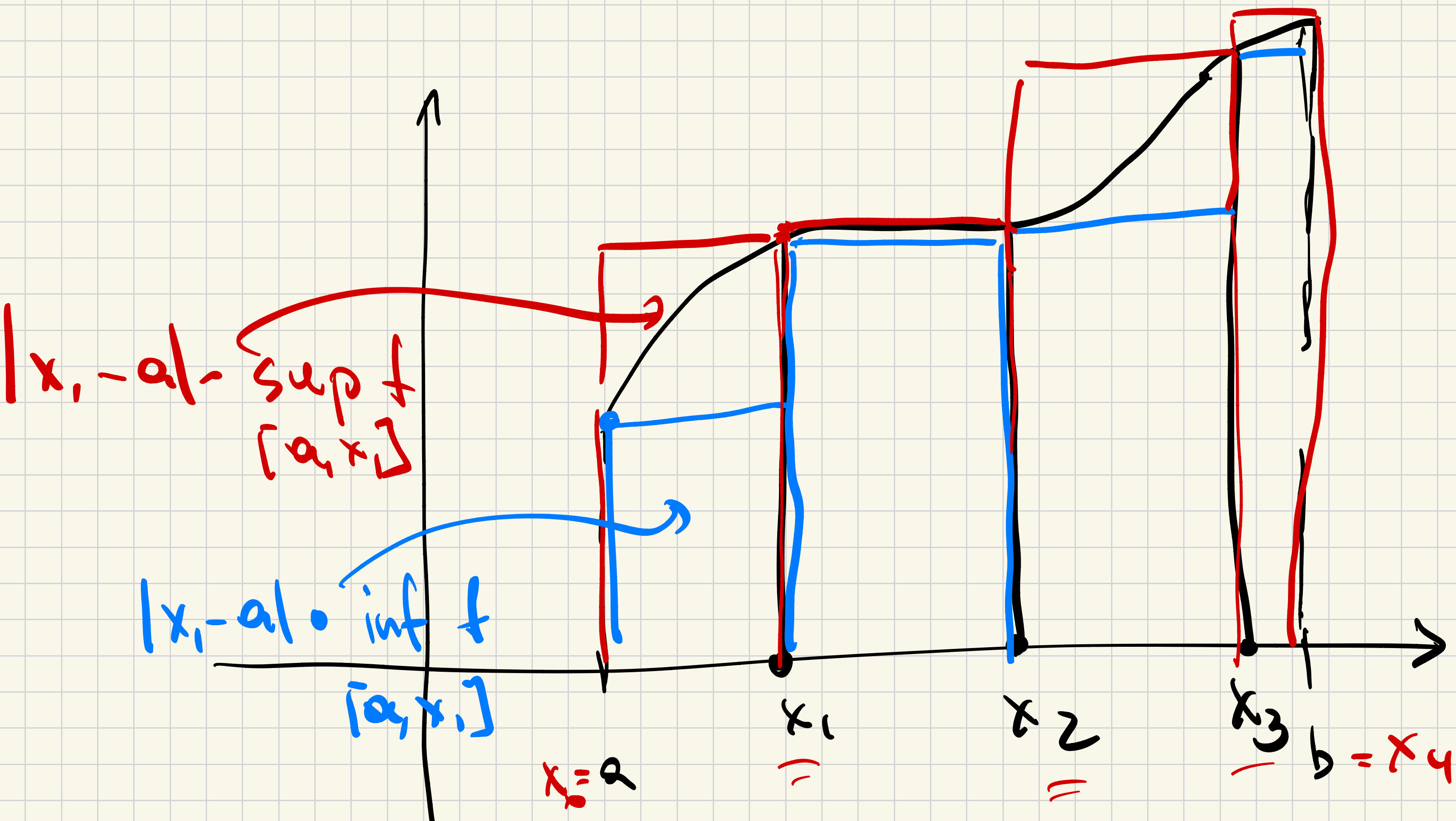




let's compute $A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$

$= f(x).$

How to define the integral?



Idea: split $[a,b]$ into smaller intervals
Aproximate area by by rectangles.

Definition Darboux sum

σ

Let $f: [a,b] \to \mathbb{R}$ be bounded and let σ be a partition of $[a,b]$ then we define

i.e. choice of $x_0 = a < x_1 < \dots < x_n = b$

$$\overline{S_\sigma} = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \quad \text{where}$$

Upper Darboux sum

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$\underline{S_\sigma} = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \quad \text{where} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Lower Darboux sum

Definition 87 Darboux integral

The upper Darboux integral of f on $[a,b]$

$$\overline{S} := \inf \left\{ \overline{S}_\sigma \;\middle|\; \sigma \text{ is a partition of } [a,b] \right\}$$

The lower Darboux integral

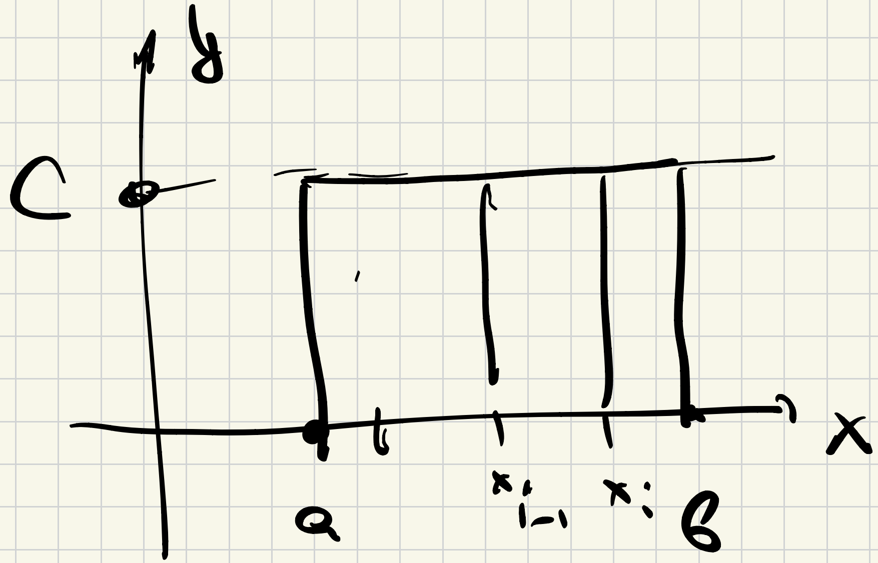
$$\underline{S} := \sup \left\{ \underline{S}_\sigma \;\middle|\; \sigma \text{ is a partition of } [a,b] \right\}$$

Definition 8.11 Let $f : [a,b] \to \mathbb{R}$ be bounded

then we say that f is integrable if $\overline{S} = \underline{S}$. We call this common value the integral of f between a and b and denote it by

$$\int_a^b f(x)\,dx.$$

Example: $f(x) = c$, $\int_a^b c \, dx$



(We should expect $c \cdot (b-a)$)

Choose any partition σ of $[a,b]$

i.e. $a = \underline{x_0 < x_1 < \ldots < x_k} = b$

· We get that $M_i = m_i = c \quad \forall\, i = 1 \ldots k$

So we get

$$\overline{S}_\sigma = \sum_{i=1}^{k} M_i \, |x_i - x_{i-1}| =$$

$$= \sum_{i=1}^{k} c \cdot |x_i - x_{i-1}| = c \sum_{i=1}^{k} |x_i - x_{i-1}|$$

$$= c \cdot (b-a)$$

Similarly we get

$$\underline{S}_{\sigma} = \sum_{i=1}^{k} m_i |x_i - x_{i-1}| =$$

$$= c \cdot \sum_{i=1}^{k} |x_i - x_{i-1}| = c(b-a)$$

$$\Rightarrow \forall \sigma \quad \overline{S}_{\sigma} = \underline{S}_{\sigma} = c(b-a) \Rightarrow \overline{S} = \underline{S} = c(b-a).$$

$\blacksquare$

Example $\quad$ let $f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Then $\quad$ For any σ and any x_i, x_{i+1}

$$\sup_{[x_i\, x_{i+1}]} f = 1 \qquad \text{by density of } \mathbb{Q}$$

$$\inf_{[x_i\, x_{i+1}]} f = 0 \qquad \text{by density of } \mathbb{R} \setminus \mathbb{Q}$$

So we get that:

$$\overline{S}_\sigma = (b-a)\cdot 1 = (b-a)$$

$$\underline{S}_\sigma = (b-a)\cdot 0 = 0$$

So $\overline{S} = b-a$ $\underline{S} = 0$

are not equal $\Rightarrow f(x) = \chi_{\mathbb{Q}}$

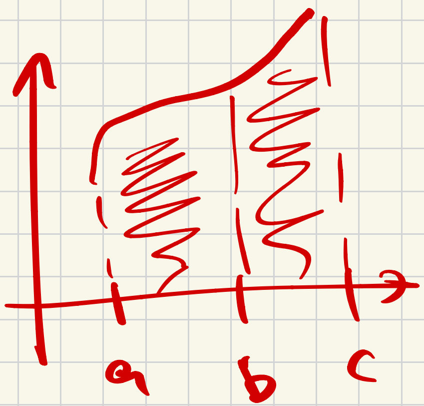
is Not Integrable. $\blacksquare$

Proposition 8.16 If $f:[a,b]\to\mathbb{R}$ is continuous then it is integrable.

First properties of integral

Proposition 8.17 Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. Then

1) If f over $[b, c]$ as integrable function then we have:



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

2) for $\alpha, \beta \in \mathbb{R}$ we get

$$\int_a^b \alpha \cdot f(x) + \beta \cdot g(x) \, dx = \alpha \cdot \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx$$

3) If $f(x) \leq g(x) \quad \forall x \in [a, b]$ then

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$

3') If f is continuous $f(x) \geq 0$ $\forall x \in [a,b]$ and

$$\int_a^b f(x) = 0 \qquad \text{then} \qquad f(x) = 0$$

4) $|f|$ is integrable on $[a,b]$ and

$$\int_a^b |f(x)| \, dx \geq \left| \int_a^b f(x) \, dx \right|$$

Analogue to $|a+b| \leq |a| + |b|$

Fundamental theorem of Calculus

Theorems 8.23 and 8.24

- Let $f: [a,b] \to \mathbb{R}$ be continuous. Then

$$F(x) = \int_a^x f(t)\,dt$$ is an anti-derivative of f

i.e. $F(x)' = f(x)$

- Let $f: [a,b] \to R$ be continuous with

anti-derivative $F(x)$. Then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$